Rational functions. It should be noted that these ideas can also be used for $f(x) = \frac{1}{x^n}$ where n is a positive integer. For $0 < a < b$, let

$$x_i^* = \sqrt[n]{\frac{(n-1)(x_{i-1}x_i)^{n-1}}{x_i^{n-2} + x_{i-1}x_i^{n-3} + x_{i-1}^2 x_i^{n-4} + \cdots + x_{i-1}^{n-3}x_i + x_{i-1}^{n-2}}}.$$

It's not too hard to show $x_{i-1} \le x_i^* \le x_i$ and $\sum_{i=1}^{k} f(x_i^*)\Delta x_i = \frac{1}{(n-1)a^{n-1}} - \frac{1}{(n-1)b^{n-1}}$.

Thus it *is* possible for our beginning calculus students to compute some definite integrals using only the definition. It does require a little finesse, and work with inequalities, but these might, in the long run, be beneficial for our students.

Summary. Students in a first semester calculus course are rarely asked to compute any integrals using only the definition of the Riemann integral. This article explains how to compute some definite integrals using only the definition and no appeal to auxiliary theorems.

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Waiting to Turn Left?

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doi:10.4169/074683410X475137

Driving home through rush hour traffic after a long day is frustrating, especially if you must make left turns at intersections without left-turn arrows. How does *your* state's Department of Transportation determine whether a specific intersection warrants a left-turn arrow?

The state of Pennsylvania uses a formula which measures the number of times vehicles potentially cross paths [**1**]. Specifically, traffic engineers have defined the *conflict factor* of a one-hour period of time as the product of the number of vehicles turning left during the hour and the number of vehicles continuing straight in the opposite direction. For example, if the northbound direction of traffic at a particular intersection is under consideration for a left-turn arrow, then the conflict between the southbound through traffic and the northbound left-turners is measured. The volume of eastbound and westbound traffic on the cross street is irrelevant. So, if 156 northbound cars turn left and 273 southbound cars continue straight over the course of one hour, the conflict factor is 42,588. The numbers are multiplied rather than added as this produces a better measure of conflict. If there are many cars turning left, but no opposing traffic, then there is no conflict and, therefore, no need for a dedicated arrow. (This resembles the standard epidemiological model where the number of new infections is proportional to the product of the number of infected and the number of susceptible. As with traffic, multiplication is appropriate because new infections are few when the number of susceptible or the number of infected is small.)

The minimum conflict factor required for the installation of a new left arrow is specified by the state. It depends on the configuration of an intersection, including features such as the number of lanes and the presence or absence of a dedicated turning lane. Suppose a road under consideration for a northbound left-turn arrow has four lanes, two northbound and two southbound, with no dedicated northbound left-turn lane. In order to qualify for a turn arrow, this intersection must have a conflict factor of at least 45,000 per hour for at least two separate (but not necessarily consecutive) hours during a normal weekday [**2**]. If the road has only one lane in each direction and there is no separate left-turn lane, the minimum threshold drops to 35,000.

Modeling real data. A traffic engineer studies an intersection by placing sensors in the road to monitor the relevant traffic volume. Let's take a look at some real data (see Table 1) captured in 15 minute intervals between 7 AM and 9 AM at an intersection in Allentown, Pennsylvania, where a busy four-lane boulevard requires a conflict factor of 45,000 for a new turn arrow.

Table 1. Hamilton Boulevard at Ott Street

Time	Westbound through	Eastbound left
7:00	66	29
7:15	53	44
7:30	100	47
7:45	91	53
8:00	89	54
8:15	85	39
8:30	89	32
8:45	113	64

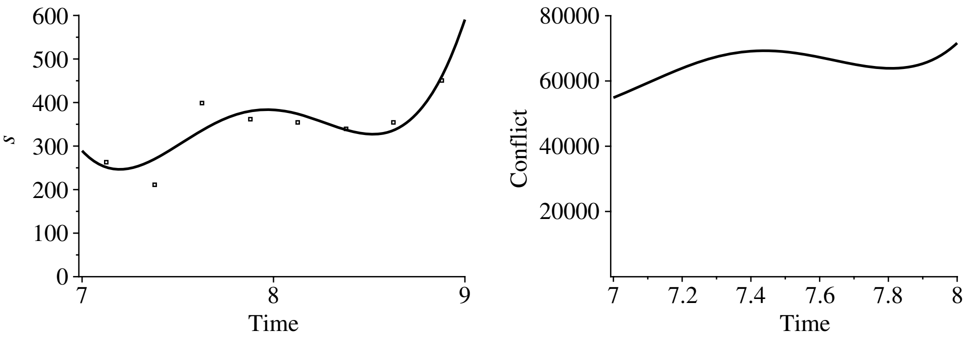
If we consider the hour from 7:00 until 8:00, we get a total of 310 cars continuing straight and 173 cars turning left, giving us a conflict factor of 53,630. The hour from 8:00 until 9:00 has a conflict factor of 71,064. Based on this data, a traffic engineer concludes that a left-turn signal is justified for the eastbound lanes of this intersection. Given this small sample of data, it is reasonable (and a good exercise for students) to use a discrete method to calculate conflict factors. However, to provide a more complete analysis, and to simplify working with larger data sets, a continuous model helps.

Let $\ell(t)$ be the number of cars turning left per hour and $s(t)$ be the number of cars continuing straight per hour for any time t where $t \in [0, 24]$. We assume these functions are continuous. A left-turn arrow is warranted if there are two values, x_1 and x_2, such that

$$F(x_i) = \left(\int_{x_i}^{x_i+1} \ell(t)\,dt\right)\left(\int_{x_i}^{x_i+1} s(t)\,dt\right) \geq 45{,}000$$

for $i = 1, 2$ where $|x_1 - x_2| \geq 1$ and $0 \leq x_1, x_2 \leq 23$. We call F the *conflict factor function*. Notice that the continuity of ℓ and s guarantees the differentiability of F.

A simple model might assume that traffic peaks only once per day, perhaps during the evening rush hour. In this case, quadratic polynomials could be used to model the traffic. A more realistic model might use Fourier series. For our data set, we converted the numbers for each fifteen minute period into an hourly rate by multiplying by four. We took this to be the rate at the midpoint of the interval. For example, since there were 66 cars going straight between 7:00 and 7:15, we have the point $(7.125, 264)$. Fourier analysis then produces $s(t) = 4(654.2 - 818.7\sin t + 13.6\cos t - 19\sin(2t) - 259.1\cos(2t))$, which is graphed in Figure 1(a) with the original data. Similar analysis produces $\ell(t)$ and the conflict factor function, $F(x)$, which is shown in Figure 1(b).



(a) Through traffic, $s(t)$

(b) Conflict factor function, $F(x)$

Figure 1. Fourier analysis of traffic data

Notice that this conflict factor function lies entirely above the threshold. Clearly, a left-turn arrow is warranted and, not surprisingly, this analysis agrees with the conclusion drawn from the discrete data. The result may not always be so obvious, so how can we solve this problem analytically? In general, we want to find two x-values which are at least an hour apart, such that $F(x) \geq 45{,}000$ where $x \in [0, 23]$. We can use calculus to help us in our search, maximizing $F(x)$ and searching for two maxima that meet these conditions.

Maximizing the conflict factor. Optimizing this function requires differentiating integrals with variable limits, so we need the fundamental theorem of calculus. Using the product rule we have

$$F'(x) = (s(x+1) - s(x))\left(\int_x^{x+1} \ell(t)\,dt\right) + (\ell(x+1) - \ell(x))\left(\int_x^{x+1} s(t)\,dt\right).$$

Since the domain of $F(x)$ is a closed interval, we look for the extreme values at the points where the first derivative is zero and at the endpoints of our interval ($x = 0$ and $x = 23$). Let S be the set of all such times whose conflict factor meets the 45,000 threshold, that is,

$$S = \{c_i \mid F(c_i) \geq 45{,}000 \text{ and either } c_i = 0,\ c_i = 23,\ \text{or } F'(c_i) = 0\}.$$

If $S = \emptyset$ then a new arrow is not necessary, as the conflict factor is never above 45,000. If $|S| \geq 2$ and $|c_i - c_j| \geq 1$ for some $c_i, c_j \in S$, then a left-turn arrow is warranted.

In the remaining cases, all elements of S are clustered within a single hour. First, suppose that all elements of S are within less than one unit of an endpoint, that is $S \subseteq [0, 1)$ or $S \subseteq (22, 23]$. If the members of S are all close to 0, then find the smallest d such that $F(d) \geq 45{,}000$ (we know such a value exists in $[0, 1)$ since $S \neq \emptyset$ and F is continuous). If $F(d+1) \geq 45{,}000$, then a left-turn arrow is warranted, otherwise it is not. To see this, suppose that $F(d+1) < 45{,}000$. Then $F(x) < 45{,}000$ for all $x \geq d+1$, otherwise there would be a member of S outside of the interval $[0, 1)$ (either a relative maximum or 23). Thus, if $F(d+1)$ does not meet the 45,000 threshold, then the threshold is not met at any time greater than $d+1$. A similar argument holds when $S \subseteq (22, 23]$. Here, we consider the largest d such that $F(d) \geq 45{,}000$ and check whether $F(d-1) \geq 45{,}000$. If it is, then a left-turn arrow is warranted, otherwise it is not.

This leaves us to consider the case where there is at least one element of S in $[1, 22]$ and all values of S are clustered within less than one hour. Notice that the conflict factor at $x = 0$ and $x = 23$ must be less than 45,000. Let c_α be the minimum of S and c_ω be the maximum of S. By the intermediate value theorem (using the definition of S and the continuity of F), there exists an $m \in (0, c_\alpha]$ such that $F(m) = 45{,}000$. Since there is no member of S smaller than c_α, we have $F(x) < 45{,}000$ when $x < m$, and, in the case where $m \neq c_\alpha$, $F(x)$ is increasing for $m \leq x \leq c_\alpha$. So, m is the unique solution to $F(x) = 45{,}000$ in the interval $(0, c_\alpha]$. Similarly, we have a unique $M \in [c_\omega, 23)$ such that $F(M) = 45{,}000$, and $F(x) < 45{,}000$ when $x > M$. If $M - m \geq 1$, then a left-turn arrow is warranted, otherwise it is not.

Concluding remarks. Modeling the data with continuous functions, and finding the zeroes of $F'(x)$ and the solutions to $F(x) = 45{,}000$, may require approximations and the wise use of technology. This is a good quantitative project for a second-semester calculus or an applied modeling course. Students will be even more interested in the project if real world data is available from their local Department of Transportation.

Summary. This article examines the rule used by the state of Pennsylvania to determine when the installation of a left-turn signal is justified. In creating a mathematical model, we encounter a natural application of the fundamental theorem of calculus.

Acknowledgments. Many thanks to Dan Hartzell for his help in obtaining PennDOT Publication 149, and Ron Penrose for providing the traffic data. We also thank the referees and editor for their careful reading and helpful suggestions.

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